evident that the curves for the highly conducting substratum are virtually identical. Thus, if either the surface impedance E_z/H_x or the wave-tilt ratio H_y/H_x is measured, the results can be interpreted in terms of the plane-wave correction factor Q. However, in the case of the highly insulating substratum, for this value of S, there is a marked departure of the curves between Δ and δ, and from the plane-wave case denoted by Q. Actually, if S is increased to 100, the curves all become completely indistinguishable for both $K = 100$ and 0.01.

Final Remark

These results, which are a sample of numerous calculations, illustrate our point that the plane-wave theory must be used with caution for interpreting electromagnetic data for a stratified earth with highly resistive sub-strata.

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Parametric Interactions Between Alfven Waves and Sonic Waves

CHARLES ELACHI

***Abstract*—Parametric interactions between a sonic pump wave and a weak Alfven wave are studied. It is shown that if the Alfven-wave velocity V_A is small relative to the sonic-wave velocity V_s there is a time-growing instability leading to the increase of the Alfven wave at the expense of the sonic wave. This phenomenon can be of importance in solar and stellar physics. For V_A large relative to V_s, the interaction is of the stop-band type.**

I. Introduction

In this communication we apply a well-developed method used in studying electromagnetic wave propagation and source radiation in space–time periodic media [1]–[5] to investigate the parametric interactions between an Alfven wave and a sonic wave. These types of interactions can occur in the laboratory or in stellar media such as the sun's atmosphere. This nonlinear problem can be linearized if we suppose the Alfven wave is weak relative to the sonic wave which modulates the plasma density in a wave-like manner. We show that a nonconvective (time-growing) instability can occur, leading to the growth of the Alfven wave at the expense of the sonic wave. The method used can be applied to a wide spectrum of astrophysical, planetary, and laboratory problems involving interactions between electromagnetic, acoustic, hydromagnetic, and space-charge waves in fixed or moving plasmas [6], [7].

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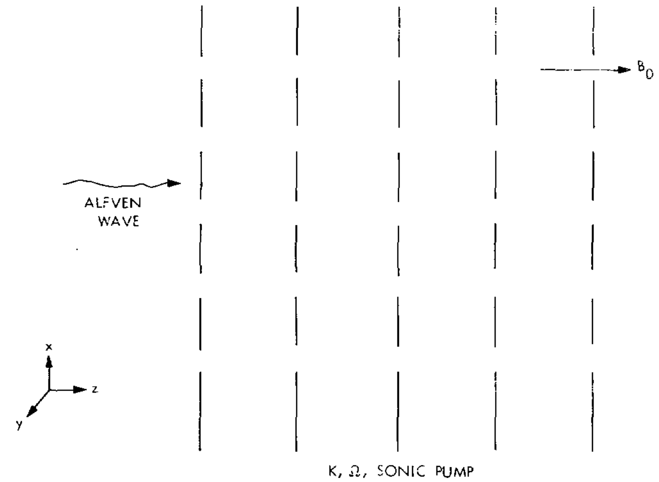

Fig. 1. Geometry of interaction. Dashed lines represent crests of sonic wave. Both sonic and Alfven waves propagate parallel to magnetic field.

II. Formulation and Discussion

The geometry considered is shown in Fig. 1. The sonic pump wave (dashed lines represent the wave crests) modulates the plasma density ρ_0 in a wave-like manner

$$\rho_0(z,t) = \rho_0[1 + \eta f(Kz - \Omega t)]$$

where $f(\xi)$ is a normalized periodic function

$$f(\xi) = \sum_{m=-\infty}^{m=+\infty} a_m e^{im\xi}$$

K and $\Omega/2\pi$ are the wavenumber and frequency of the sonic pump, $V_s = \Omega/K$ is the sonic velocity, and $\eta < 1$ corresponds to the strength of the pump and is supposed to be small enough so we can neglect terms of the second order (or higher) in η. The appropriate equations governing the Alfven wave behavior are [8]

$$\rho_0(z,t)\left[\frac{\partial}{\partial t} + U(z,t)\frac{\partial}{\partial z}\right]\boldsymbol{v} + \frac{\boldsymbol{B}_0}{4\pi} \times \nabla \times \boldsymbol{b} = 0 \quad (1)$$

$$\frac{\partial \boldsymbol{b}}{\partial t} - \nabla \times (\boldsymbol{v} \times \boldsymbol{B}_0) = 0 \quad (2)$$

$$\boldsymbol{U} = \eta V_s f(Kz - \Omega t)\boldsymbol{e}_z$$

where $\boldsymbol{U}$ is the particles' velocity associated with the sonic wave; $\boldsymbol{b}$ and $\boldsymbol{v}$ are the magnetic field and particles' velocity associated with the Alfven wave, and we took into consideration that this wave is transverse and does not generate change in the plasma density. $\boldsymbol{B}_0 = B_0\boldsymbol{e}_z$ is a static magnetic field. Because of the space–time periodicity, the wave solution consists of an infinite number of space and time harmonics (Floquet form)

$$\boldsymbol{v} = \boldsymbol{e}_t \sum_{n=-\infty}^{n=+\infty} v_n \exp(i\kappa_n z - i\omega_n t)$$

$$\boldsymbol{b} = \boldsymbol{e}_t \sum_{n=-\infty}^{n=+\infty} b_n \exp(i\kappa_n z - i\omega_n t)$$

$$\kappa_n = \kappa + nK, \qquad \omega_n = \omega + n\Omega$$

where $\boldsymbol{e}_t$ is the transverse unit vector, v_n and b_n correspond to the generated harmonics, $\omega/2\pi$ is the principal Alfven frequency, and

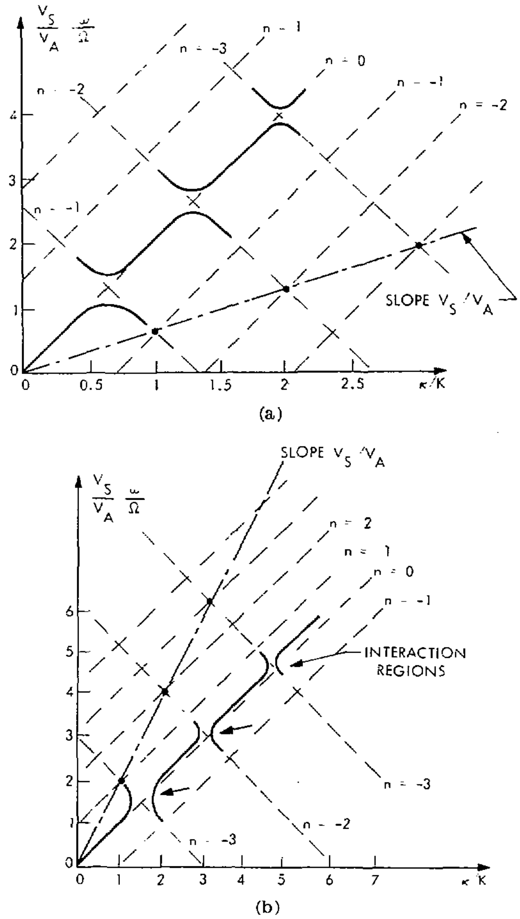

Fig. 2. Brillouin diagram: broken line—$\eta = 0$: solid line—$\eta \neq 0$. (a) $V_A > V_s$. (b) $V_A < V_s$.

κ is a wavenumber which will be determined from the dispersion equation. Replacing the quantities in (1) and (2) by their Floquet form, equating the terms with the same frequency, and solving the resulting equations, we obtain

$$b_n = -\frac{\kappa_n}{\omega_n} B_0 v_n$$

$$D_n V_n + \eta \sum_{j=-\infty}^{j=+\infty} a_{n-j} V_j = 0 \qquad (3)$$

where

$$V_n = (\omega_n - V_s \kappa_n) v_n$$

$$D_n = 1 - \frac{\kappa_n}{\omega_n} \frac{\kappa_n V_A{}^2 - \omega_n V_s}{\omega_n - \kappa_n V_s}$$

$$V_A = B_0/(4\pi\rho_0)^{1/2} = \text{Alfven velocity.}$$

The solution of the system of equations (3) gives the relative amplitudes V_n/V_0, and the nontriviality condition (system's determinant = 0) gives the dispersion relation which determines the Brillouin diagram. The convergence of the Floquet sum (3) can be determined by applying the Poincaré convergence theorem [1]. For a sinusoidal modulation the sum would converge if $|\lim_{n\to\infty} D_n| > 1$. This condition is always satisfied, and therefore the sum converges.

Exact solutions would require extensive numerical calculation, but the limit $\eta \ll 1$ can be studied analytically and gives a good idea of the general solution. For $\eta = 0$, (3) reduces to

$$D_n = 0 \Longrightarrow \omega_n = \pm V_A \kappa_n. \qquad (4)$$

The corresponding Brillouin diagram is a series of lines of slope $\pm V_A$ centered at $O_n = -nK, -n\Omega$ (see dashed lines in Fig. 2), but only the harmonic $n = 0$ has any physical meaning. For $\eta \neq 0$, strong interactions occur near the intersection points because of the phase-matched coupling between the harmonics. In the limit $\eta \ll 1$, Taylor series development could be applied to analytically determine the behavior near the interaction regions.

Let us consider the two harmonics $n = 0$ and $n = -1$ (the following is valid for any interaction region between two successive harmonics). For $\eta = 0$, the corresponding Brillouin diagrams intersect at $\kappa^0 = K/2 + \Omega/2V_A$ and $\omega^0 = \Omega/2 + KV_A/2$. For $\eta \neq 0$, but small, the solution near the interaction region can be written as $\kappa = \kappa^0(1 + \eta r)$ and $\omega = \omega^0(1 + \eta s)$ and all harmonics other than $n = 0$ and -1 can be neglected. Therefore, (3) reduces to (for $f(\xi) = \cos\xi$, i.e., $a_{\pm 1} = \frac{1}{2}$ and $a_m = 0$, otherwise)

$$D_0 D_{-1} = \frac{\eta^2}{4}. \qquad (5)$$

Replacing κ and ω by their Taylor series expansion, (5) reduces to

$$r^2 - s^2 = \frac{V_s - V_A}{V_s + V_A}\left[\frac{V_s - V_A}{4V_A}\right]^2. \qquad (6)$$

The resulting conclusions are as follows.

1) For $V_s < V_A$ (Fig. 2(a)), (6) corresponds to a stop-band interaction where ω is complex (s is real and r is imaginary), and the generated harmonics can be relatively large

$$\frac{V_{-1}}{V_0} \approx \left[\frac{V_A - V_s}{V_A + V_s}\right]^{1/2}$$

near the first interaction region.

2) For $V_s > V_A$ (Fig. 2(b)), (6) corresponds to an unstable time-growing interaction where ω is complex (s is imaginary and r is real). This type of instability means that if the magnetoplasma supports a strong sonic wave, a small Alfven disturbance would grow by extracting power from the sonic wave until some nonlinear processes stop its growth. The rate of growth for a first-order interaction is determined by

$$\max\left[\frac{\mathrm{Im}\,\omega}{\omega^0}\right] = \eta\,\frac{V_s - V_A}{4V_A}\left[\frac{V_s - V_A}{V_s + V_A}\right]^{1/2}$$

and the unstable Alfven frequencies correspond to the intersection points between the different space harmonics. The intersection between the nth and mth harmonic gives

$$\left.\begin{aligned} \omega_n &= V_A \kappa_n \\ \omega_m &= -V_A \kappa_m \end{aligned}\right\} \rightarrow \omega = \frac{\Omega}{2}\left[(n - m)\frac{V_A}{V_s} - (n + m)\right]$$

where $n,m = 0, \pm 1, \cdots$.

III. Conclusion

The well-developed techniques for the study of electromagnetic waves in space–time periodic media were applied for the study of the interaction between sonic and Alfven waves. The results of this communication can be applied to many physical phenomena, especially for the generation of Alfven waves in the atmosphere of the sun.

Strong sound waves can exist in the sun's chromosphere, and the sound velocity, which is usually of the order of 10 km/s, can exceed the Alfven velocity, which changes from $\approx$0.2 km/s in the photosphere to 1000 km/s in the chromosphere [9]. Therefore, it is possible that in some regions of the sun's atmosphere (where $V_s > V_A$) time-growing Alfven waves are generated at the expense of sonic waves.

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An Application of One-Dimensional Inverse-Scattering Theory for Inhomogeneous Regions

A. K. JORDAN AND H. N. KRITIKOS

***Abstract*—One-dimensional inverse-scattering theory is applied to the study of the reflection of electromagnetic waves from an inhomogeneous region having a refractive index $n(x)=[1-(1/k^2)q(x)]^{1/2}$ where $k=2\pi/\lambda$, and λ is the free-space wavelength. The exact refractive index profile is obtained that will produce a reflection coefficient in which the frequency dependence is described by the Butterworth approximation.**

I. Introduction

Inverse scattering theory is concerned with the reconstruction of the physical characteristics of an unknown target from the knowledge of the scattering data, i.e., the incident and scattered electromagnetic waves. Analogously, the synthesis problem is concerned with the design of a physical system in which the scattering characteristics are prescribed to have a given functional form.

As with direct scattering theory, there are several approaches to inverse scattering theory, e.g., physical optics or Fourier transform techniques [1], geometrical optics techniques [2], extended boundary condition methods [3], and "exact" or spectral methods [4]. A synthesis procedure when the reflection coefficient is prescribed to be a rational function of frequency has been developed by Kay [5] from the basic mathematical theory of Gelfand and Levitan [4], [6]. Such a synthesis technique is also of interest in the design of nonuniform transmission lines [7].

The purpose of the present communication is to give an application of spectral one-dimensional inverse scattering theory when the reflection coefficient is specified to be a Butterworth function of frequency [9]. The physical model considered is useful in the study of the scattering of millimeter waves by semiconductor surfaces [10].

II. Definition of the Problem

The physical model, which is considered here, is the scattering of electromagnetic waves from an inhomogeneous dispersive medium. An example of such a region is an intrinsic semiconductor whose charge carrier density $N(x)$ is a function of the space dimension x [11]. The reflection coefficient $b(k)$ is assumed to be known for all (normalized) frequencies, $0 \leq k < \infty$, and to be a prescribed function of frequency. Thus for the purposes of this brief communication, the experimental and data-reduction problems of obtaining the functional form for $b(k)$ are excluded.

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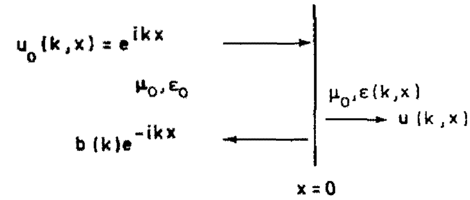

Fig. 1. Geometry for scattering by inhomogeneous semiconductor region.

A plane wave of unit amplitude $u_0 = e^{ikx}$, is normally incident upon an inhomogeneous and dispersive region in the right half-plane, $0 \leq x < \infty$, as shown in Fig. 1. A reflected wave $b(k)e^{-ikx}$, is produced, where $b(k)$ is the complex reflection coefficient. The factor $e^{-i\omega t}$ is suppressed throughout this discussion. In the right half-plane the wave function, $u(k,x)$ satisfies the differential equation

$$\frac{d^2}{dx^2}u(k,x) + k^2n^2(k,x)u(k,x) = 0 \tag{1}$$

where $n(k,x)$ is the index of refraction of the region, and

$$n^2(k,x) = 1 - \frac{1}{k^2}q(x) \tag{2}$$

where $q(x)$ is the scattering potential. Dispersion relations of this type in practice can be observed in intrinsic semiconductors where

$$q(x) = \frac{e_0^2}{\epsilon_0 m_0^2 c^2}N(x)$$

where e_0 is the charge, m_0 is the effective mass of the charge carrier, ϵ_0 is the permittivity of free space, and $N(x)$ is the charge carrier density profile. The dispersion relation (2) can be obtained by considering the scattering of electromagnetic waves from a solid-state plasma [12]. The problem, which is considered here, is to determine $N(x)$ given that $b(k)$, or $|b(k)|^2$, is a rational function of frequency that exhibits bandpass characteristics.

III. Inverse Scattering Theory

Gelfand and Levitan have examined [6] the one-dimensional wave equation

$$\frac{d^2}{dx^2}u(k,x) + k^2u(k,x) = q(x)u(k,x), \qquad x \geq 0 \tag{3}$$

which is obtained from (1) with the index of refraction of (2). The solution of (3) can be represented as [5], [6]

$$u(k,x) = u_1(k,x) + \int_{-y}^{x} K(x,\xi)u_1(k,\xi)\,d\xi \tag{4}$$

where $u_1(k,x) = u_0(k,x) + b(k)e^{-ikx}$, $x \leq 0$.

By considering the spectral properties of the boundary value problem, Gelfand and Levitan have shown that the kernel function $K(x,y)$ can be uniquely determined from the solution of the integral equation

$$B(x,y) + K(x,y) + \int_{-y}^{x} K(x,\xi)B(\xi + y)\,d\xi = 0 \tag{5}$$

where $B(x)$ is the Cauchy principal value of the inverse Fourier transformation of the reflection coefficient. The kernel function has the properties

$$K(x,-x) = 0 \tag{6}$$

$$\frac{d}{dx}K(x,x) = \frac{1}{2}q(x). \tag{7}$$